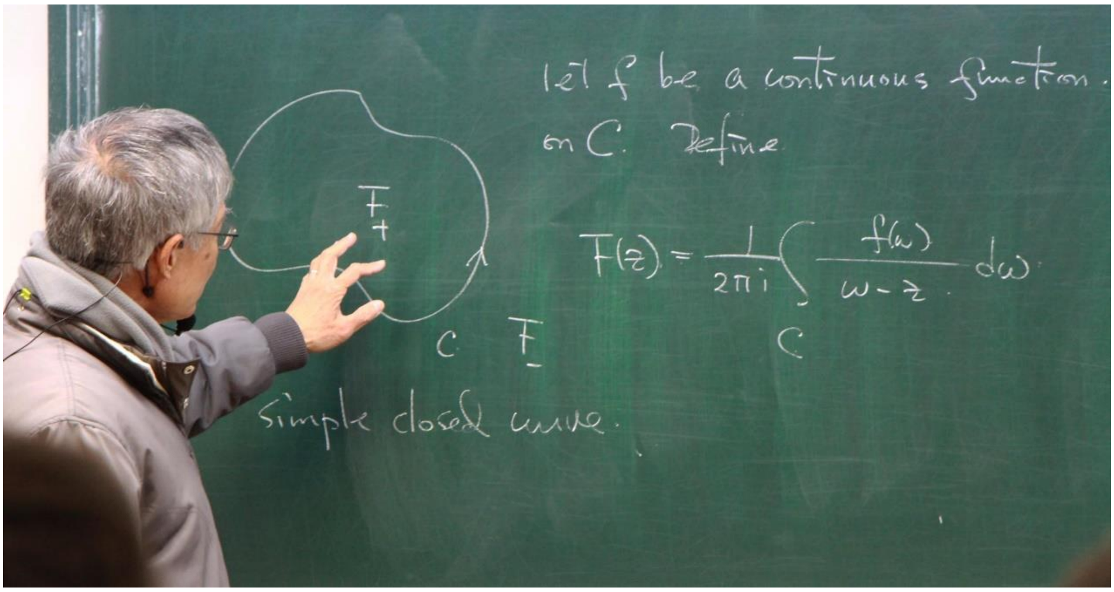
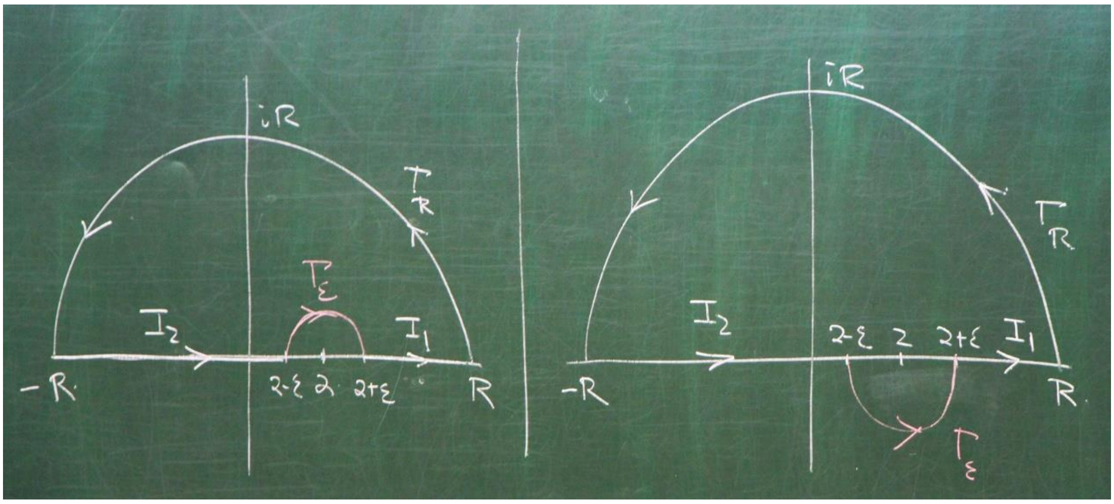
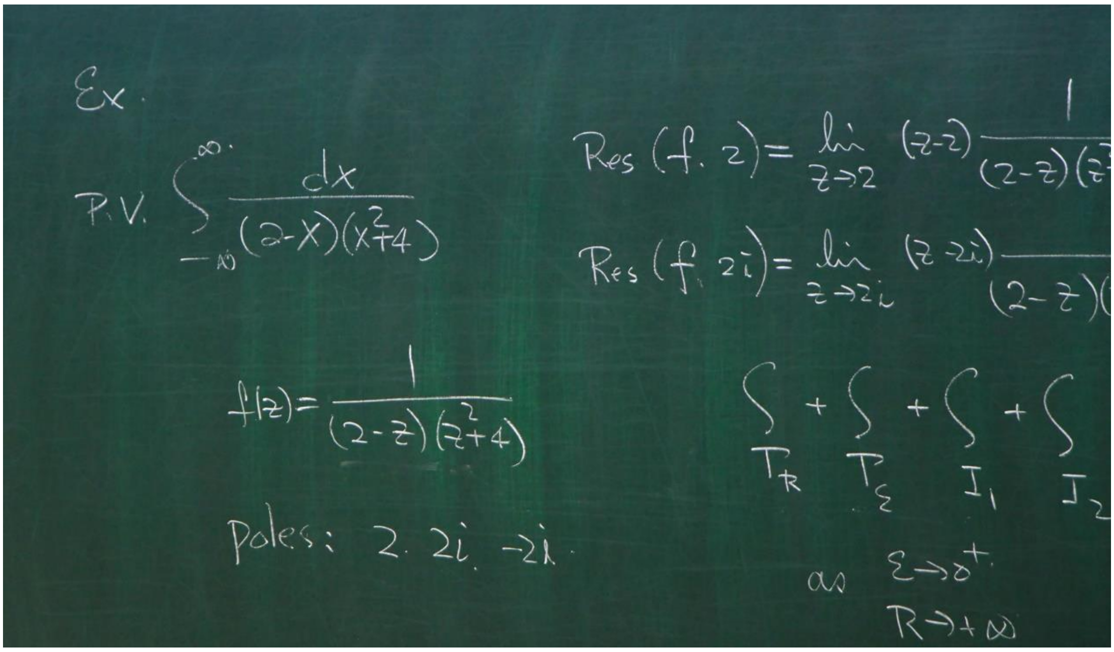
【10920 程守慶教授複變數函數論 / 第 8 堂版書】



let f be a continuous function on C. Define

$$F(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}\,dw.$$





Ex.

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{dx}{(2-x)(x^2+4)}$$

$$\text{Res}\,(f.\ 2) = \lim_{z\to 2}\ (z-2)\frac{1}{(2-z)(z}$$

$$\text{Res}\,(f,\ 2i) = \lim_{z\to 2i}\ (z-2i)\frac{}{(2-z)(}$$

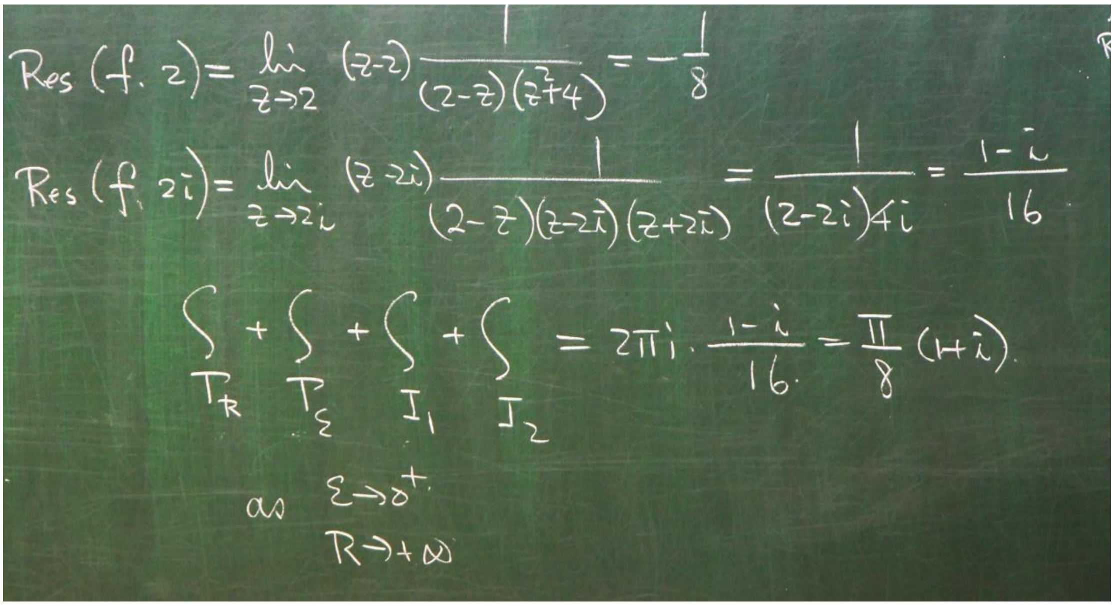
$$f(z) = \frac{1}{(2-z)(z^2+4)}$$

poles: $2.\ 2i.\ -2i.$

$$\int_{T_R} + \int_{T_\varepsilon} + \int_{I_1} + \int_{I_2}$$

as $\varepsilon \to 0^+$

$R \to +\infty$



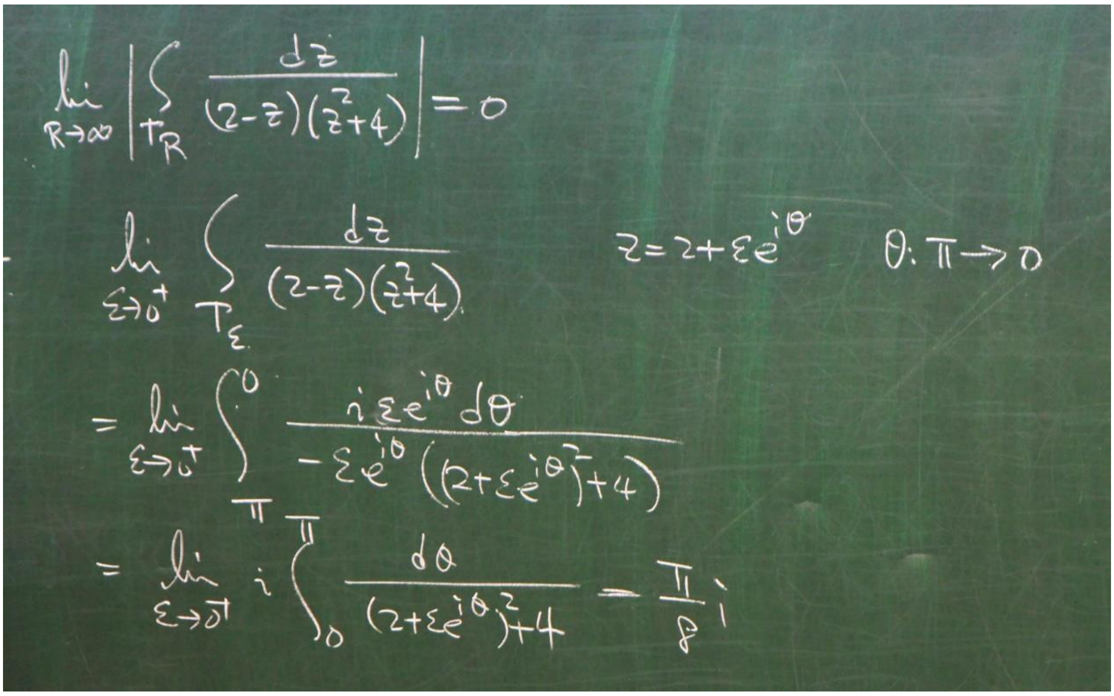
$$\text{Res}\,(f.\ 2) = \lim_{z\to 2}\ (z-2)\frac{1}{(2-z)(z^2+4)} = -\frac{1}{8}$$

$$\text{Res}\,(f,\ 2i) = \lim_{z\to 2i}\ (z-2i)\frac{1}{(2-z)(z-2i)(z+2i)} = \frac{1}{(2-2i)4i} = \frac{1-i}{16}$$

$$\int_{T_R} + \int_{T_\varepsilon} + \int_{I_1} + \int_{I_2} = 2\pi i \cdot \frac{1-i}{16} = \frac{\pi}{8}(1+i).$$

as $\varepsilon \to 0^+$

$R \to +\infty$

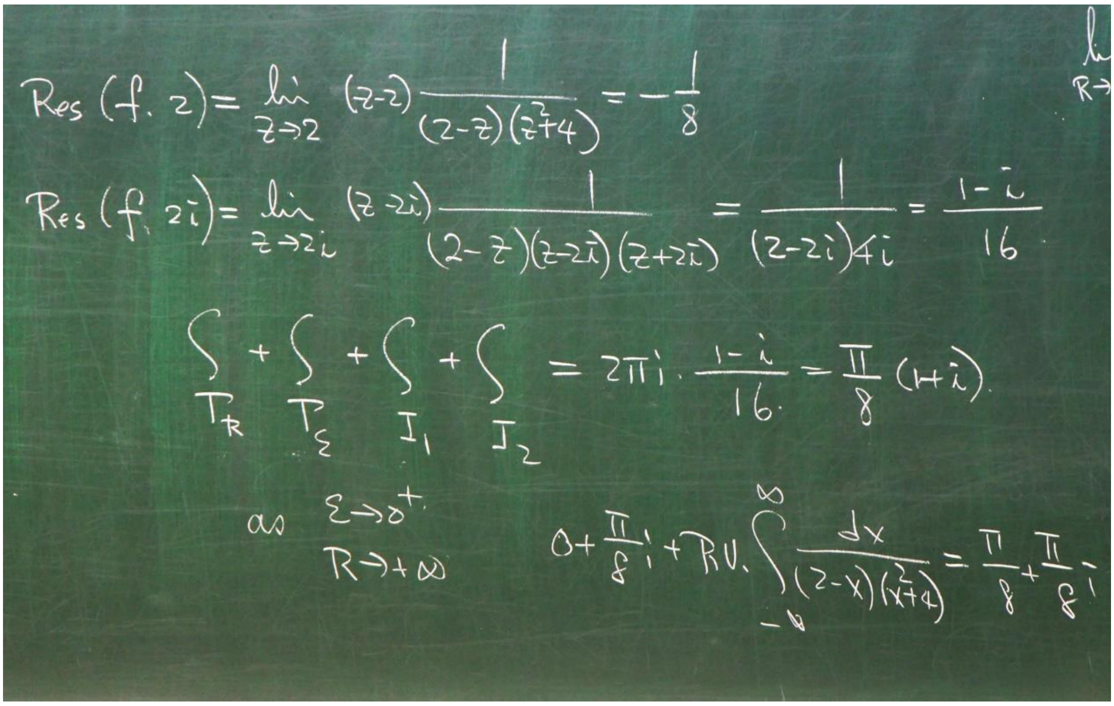


$$\lim_{R\to\infty}\left|\int_{T_R}\frac{dz}{(2-z)(z^2+4)}\right|=0$$

$$\lim_{\varepsilon\to0^+}\int_{T_\varepsilon}\frac{dz}{(2-z)(z^2+4)} \qquad z=2+\varepsilon e^{i\theta} \qquad \theta:\pi\to0$$

$$=\lim_{\varepsilon\to0^+}\int_{\pi}^{0}\frac{i\varepsilon e^{i\theta}\,d\theta}{-\varepsilon e^{i\theta}\left((2+\varepsilon e^{i\theta})^2+4\right)}$$

$$=\lim_{\varepsilon\to0^+} i\int_{0}^{\pi}\frac{d\theta}{(2+\varepsilon e^{i\theta})^2+4}=\frac{\pi}{8}i$$



$$\lim_{R\to}$$

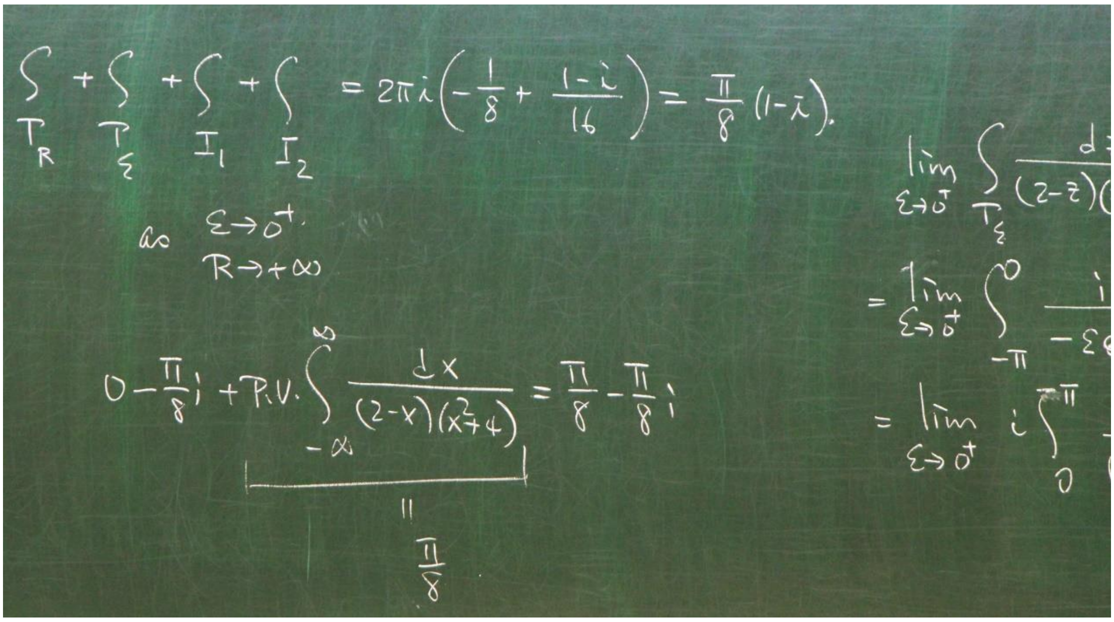
$$\operatorname{Res}(f,2)=\lim_{z\to2}(z-2)\frac{1}{(2-z)(z^2+4)}=-\frac{1}{8}$$

$$\operatorname{Res}(f,2i)=\lim_{z\to2i}(z-2i)\frac{1}{(2-z)(z-2i)(z+2i)}=\frac{1}{(2-2i)4i}=\frac{1-i}{16}$$

$$\int_{T_R}+\int_{T_\varepsilon}+\int_{I_1}+\int_{I_2}=2\pi i\cdot\frac{1-i}{16}=\frac{\pi}{8}(1+i)$$

as $\varepsilon\to0^+$, $R\to+\infty$

$$0+\frac{\pi}{8}i+\text{P.V.}\int_{-\infty}^{\infty}\frac{dx}{(2-x)(x^2+4)}=\frac{\pi}{8}+\frac{\pi}{8}i$$



$$\int_{T_R} + \int_{T_\varepsilon} + \int_{I_1} + \int_{I_2} = 2\pi i\left(-\frac{1}{8} + \frac{1-i}{16}\right) = \frac{\pi}{8}(1-i).$$

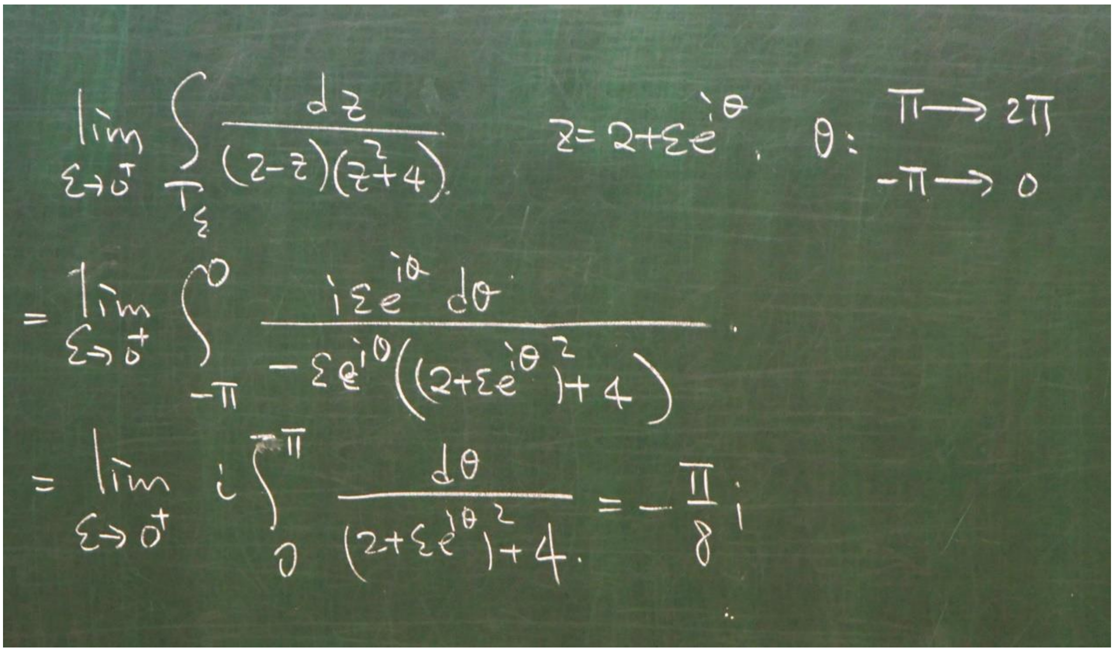
as $\varepsilon \to 0^+$, $R \to +\infty$

$$0 - \frac{\pi}{8}i + \underbrace{\text{P.V.}\int_{-\infty}^{\infty}\frac{dx}{(2-x)(x^2+4)}}_{\frac{\pi}{8}} = \frac{\pi}{8} - \frac{\pi}{8}i$$

$$\lim_{\varepsilon\to0^+}\int_{T_\varepsilon}\frac{d}{(2-z)(}$$

$$= \lim_{\varepsilon\to0^+}\int_{-\pi}^{0}\frac{i}{-\varepsilon}$$

$$= \lim_{\varepsilon\to0^+} i\int_{0}^{\pi}$$

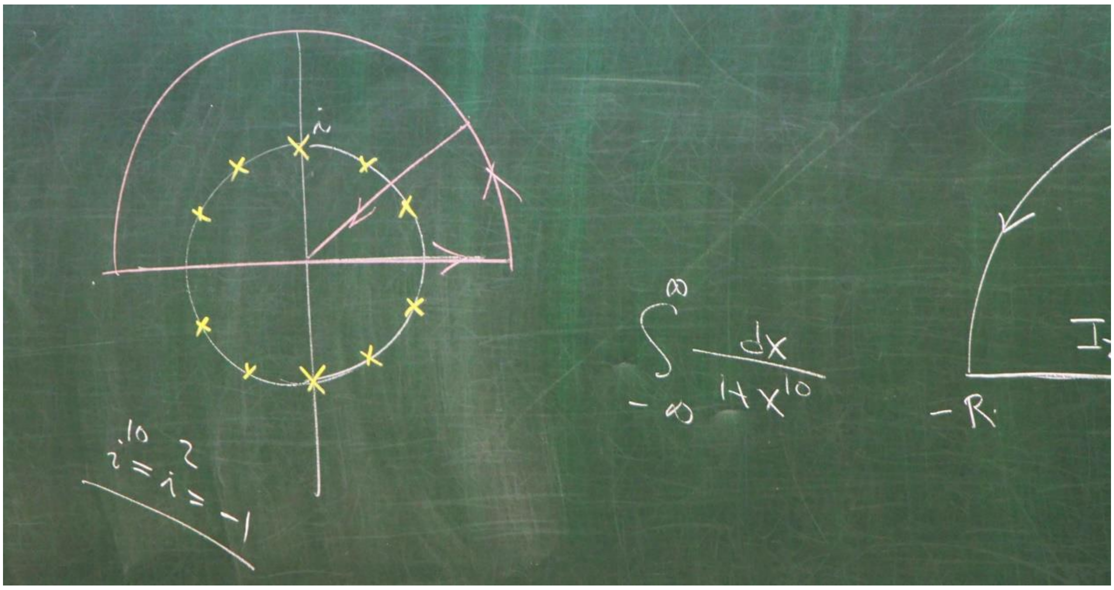


$$\lim_{\varepsilon\to0^+}\int_{T_\varepsilon}\frac{dz}{(2-z)(z^2+4)}.$$

$z = 2+\varepsilon e^{i\theta}$. θ: $\pi \longrightarrow 2\pi$
$-\pi \longrightarrow 0$

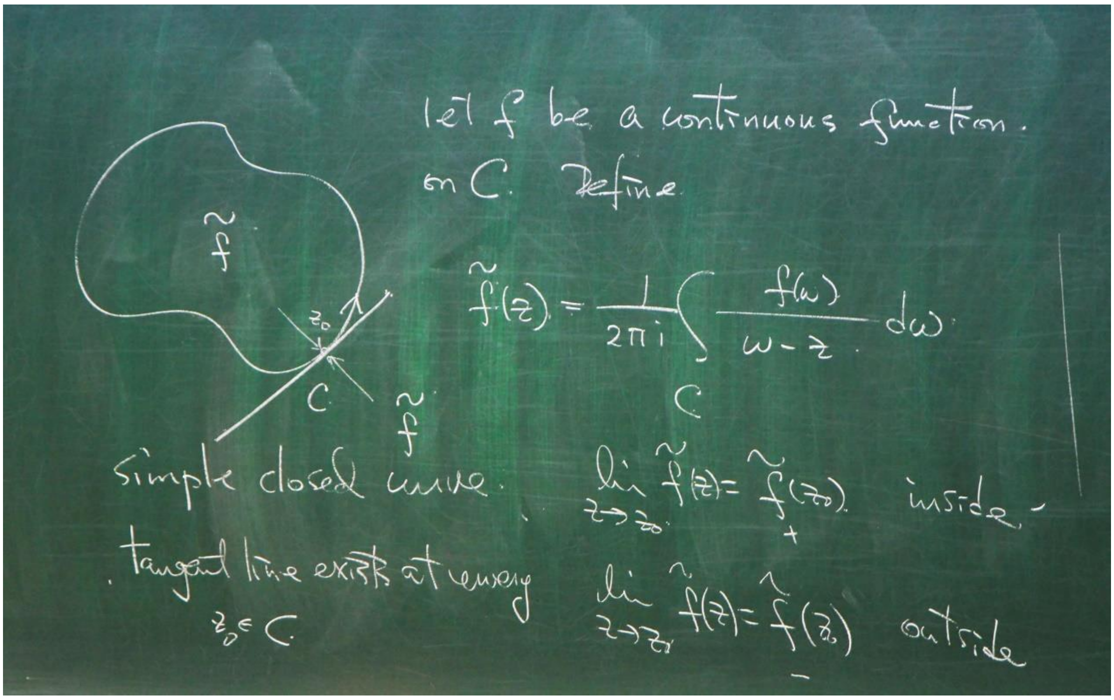
$$= \lim_{\varepsilon\to0^+}\int_{-\pi}^{0}\frac{i\varepsilon e^{i\theta}\,d\theta}{-\varepsilon e^{i\theta}\left((2+\varepsilon e^{i\theta})^2+4\right)}.$$

$$= \lim_{\varepsilon\to0^+} i\int_{0}^{\pi}\frac{d\theta}{(2+\varepsilon e^{i\theta})^2+4.} = -\frac{\pi}{8}i$$



$$\int_{-\infty}^{\infty} \frac{dx}{1+x^{10}}$$

$$i^{10} = i^{2} = -1$$



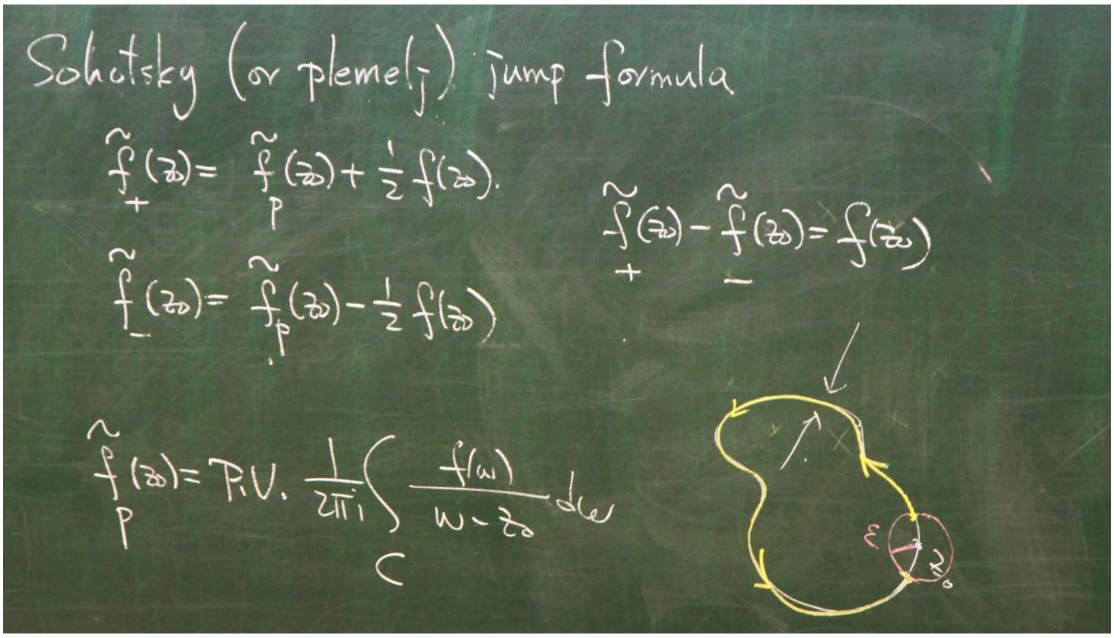
let f be a continuous function.
on C. Define

$$\tilde{f}(z) = \frac{1}{2\pi i}\int_{C} \frac{f(\omega)}{\omega - z}\, d\omega$$

simple closed curve.
tangent line exists at every $z_0 \in C$.

$$\lim_{z\to z_0} \tilde{f}(z) = \tilde{f}_{+}(z_0) \quad \text{inside}$$

$$\lim_{z\to z_0} \hat{f}(z) = \hat{f}_{-}(z_0) \quad \text{outside}$$



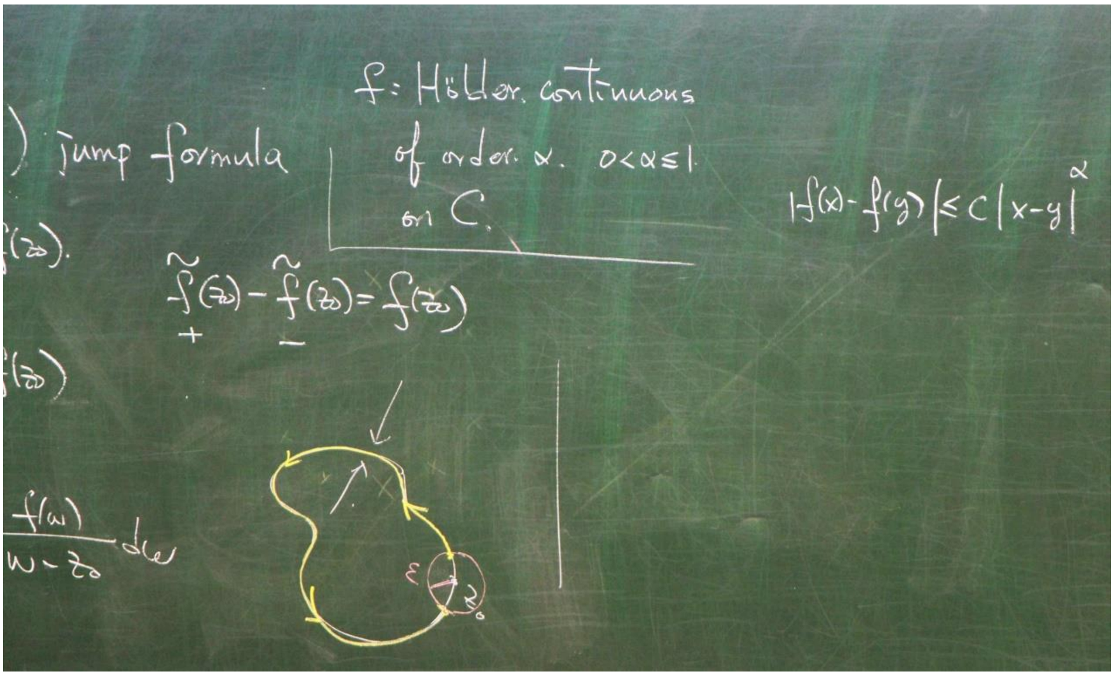
Sohotsky (or plemelj) jump formula

$$\tilde{f}_{+}(z_0) = \tilde{f}_{p}(z_0) + \frac{1}{2} f(z_0).$$

$$\tilde{f}_{-}(z_0) = \tilde{f}_{p}(z_0) - \frac{1}{2} f(z_0)$$

$$\tilde{f}_{+}(z_0) - \tilde{f}_{-}(z_0) = f(z_0)$$

$$\tilde{f}_{p}(z_0) = \text{P.V.}\ \frac{1}{2\pi i}\int_{C} \frac{f(w)}{w - z_0}\, dw$$

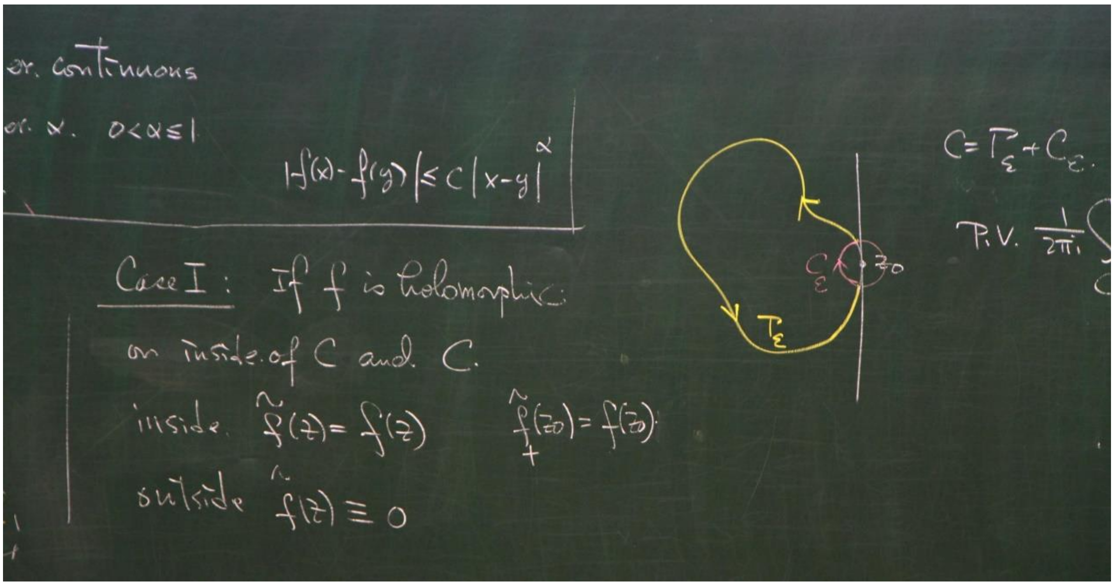


) jump formula

$$\tilde{f}_{+}(z_0) - \tilde{f}_{-}(z_0) = f(z_0)$$

f: Hölder continuous
of order α. $0 < \alpha \leq 1$
on C.

$$|f(x) - f(y)| \leq C|x - y|^{\alpha}$$



er. continuous

or. α. $0<\alpha\leq 1$

$$|f(x)-f(y)|\leq C|x-y|^{\alpha}$$

Case I : If f is holomorphic:

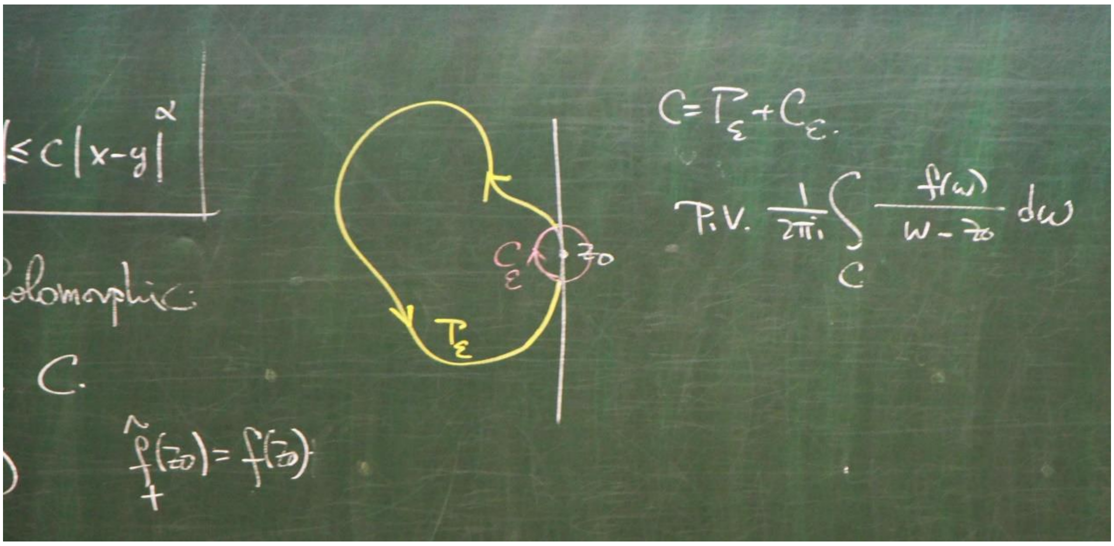
on inside of C and C.

inside $\tilde{f}(z)=f(z)$ $\quad \tilde{f}_+(z_0)=f(z_0)$

outside $\tilde{f}(z)\equiv 0$

$C=\Gamma_\varepsilon+C_\varepsilon$.

P.V. $\frac{1}{2\pi i}\int$



$\leq C|x-y|^{\alpha}$

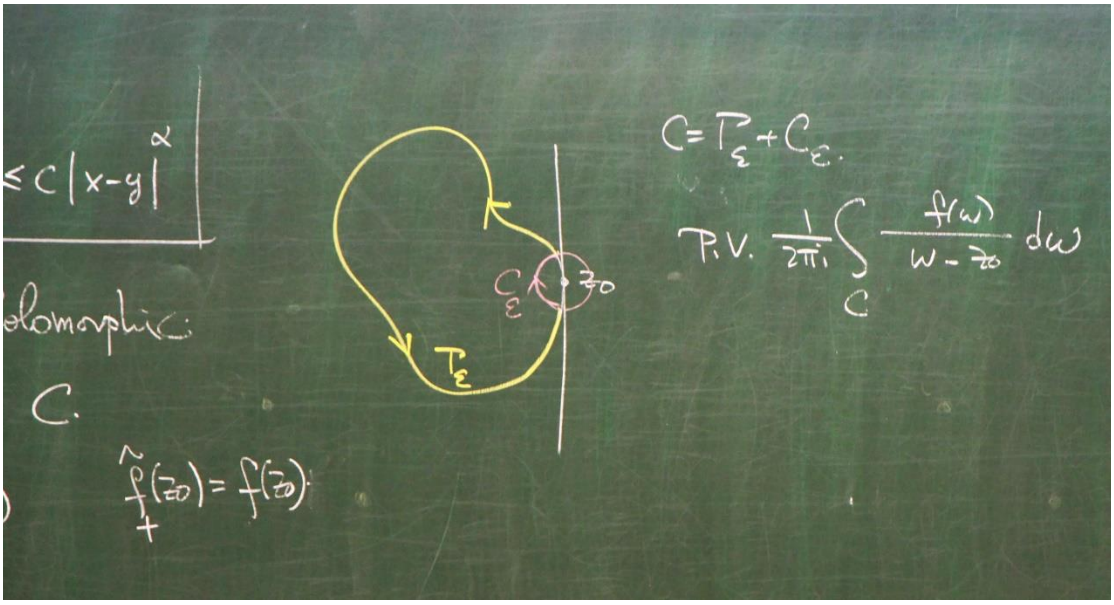
olomorphic:

C.

$\tilde{f}_+(z_0)=f(z_0)$

$C=\Gamma_\varepsilon+C_\varepsilon$.

$$\text{P.V. } \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z_0}\,dw$$



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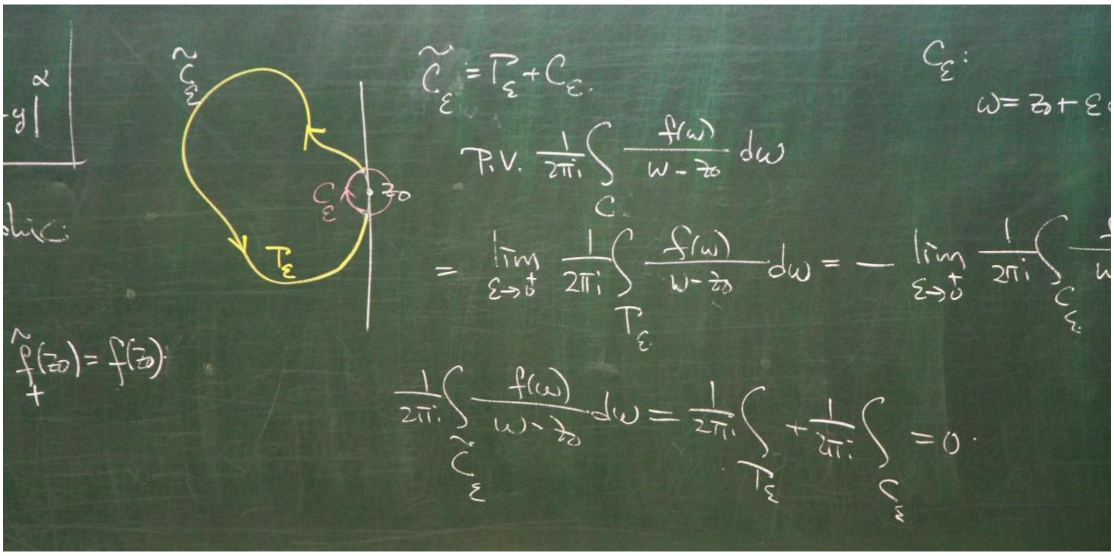


$\tilde{f}_+(z_0) = f(z_0)$.

$\tilde{C}_\varepsilon = T_\varepsilon + C_\varepsilon$.

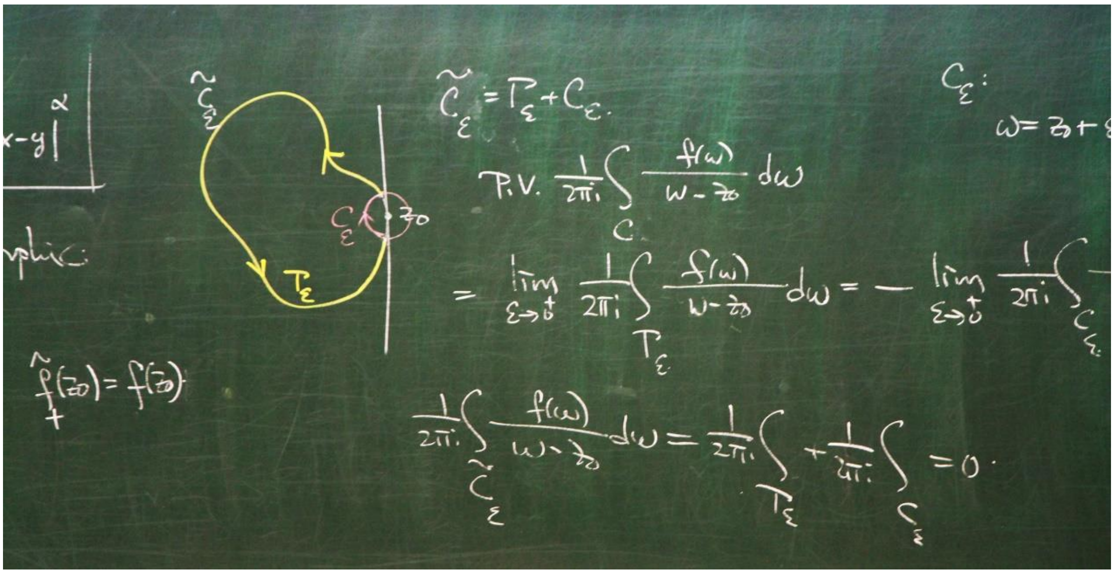
C_ε:

$\omega = z_0 + \varepsilon$

$$\text{P.V. } \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z_0}\,dw$$

$$= \lim_{\varepsilon\to 0^+} \frac{1}{2\pi i}\int_{T_\varepsilon} \frac{f(w)}{w-z_0}\,dw = -\lim_{\varepsilon\to 0^+} \frac{1}{2\pi i}\int_{C_\varepsilon}$$

$$\frac{1}{2\pi i}\int_{\tilde{C}_\varepsilon} \frac{f(\omega)}{\omega-z_0}\,d\omega = \frac{1}{2\pi i}\int_{T_\varepsilon} + \frac{1}{2\pi i}\int_{C_\varepsilon} = 0.$$


$\tilde{f}_+(z_0) = f(z_0)$.

$\tilde{C}_\varepsilon = T_\varepsilon + C_\varepsilon$.

C_ε:

$\omega = z_0 +$

$$\text{P.V. } \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z_0}\,dw$$

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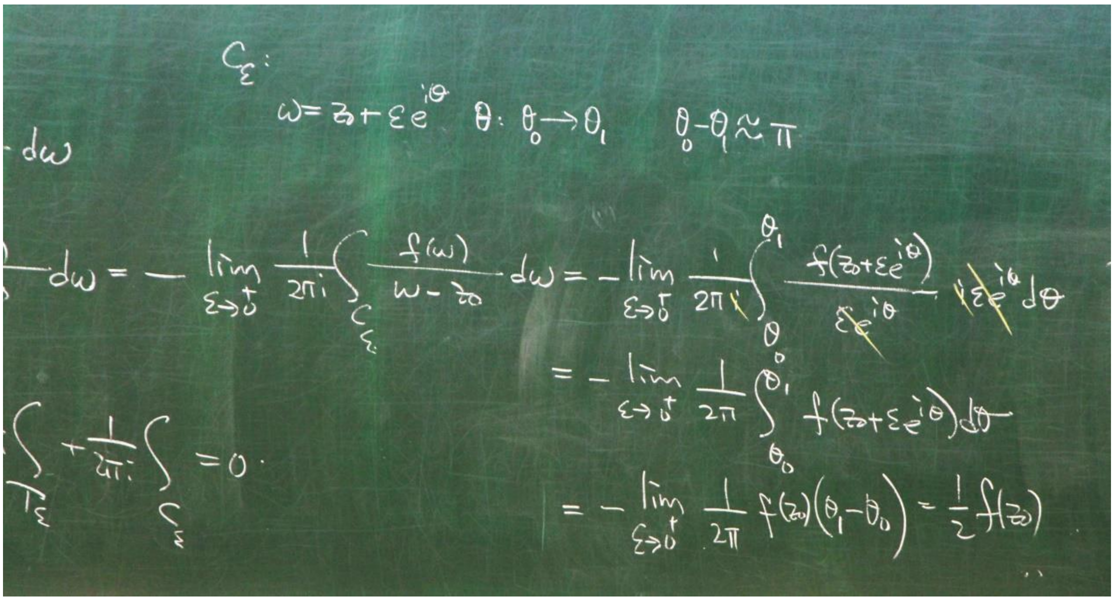


C_ε:

$\omega = z_0 + \varepsilon e^{i\theta}$ $\theta: \theta_0 \longrightarrow \theta_1$ $\theta_0 - \theta_1 \approx \pi$

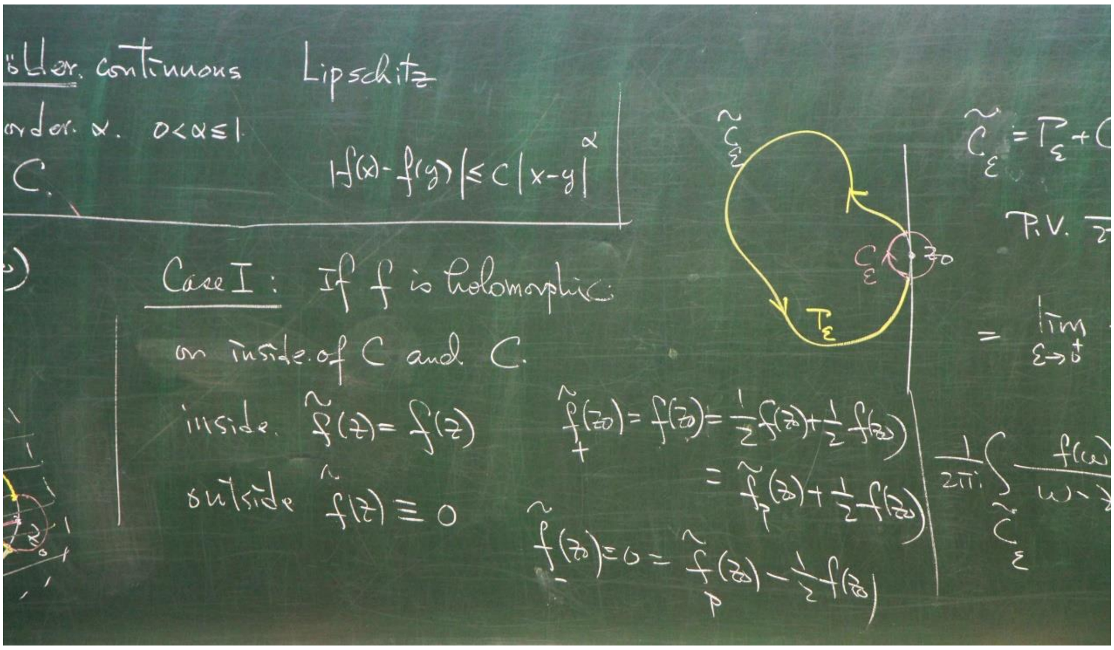
dw

$$dw = -\lim_{\varepsilon\to 0^+} \frac{1}{2\pi i}\int_{C_\varepsilon} \frac{f(\omega)}{w-z_0}\,dw = -\lim_{\varepsilon\to 0^+} \frac{1}{2\pi \cancel{i}}\int_{\theta_0}^{\theta_1} \frac{f(z_0+\varepsilon e^{i\theta})}{\cancel{\varepsilon e^{i\theta}}}\,\cancel{i\varepsilon e^{i\theta}}\,d\theta$$

$$= -\lim_{\varepsilon\to 0^+} \frac{1}{2\pi}\int_{\theta_0}^{\theta_1} f(z_0+\varepsilon e^{i\theta})\,d\theta$$

$$= -\lim_{\varepsilon\to 0^+} \frac{1}{2\pi} f(z_0)(\theta_1-\theta_0) = \frac{1}{2} f(z_0)$$

$$\int_{T_\varepsilon} + \frac{1}{2\pi i}\int_{C_\varepsilon} = 0.$$



ölder. continuous Lipschitz

order. α. $0<\alpha\le 1$

C.

$$|f(x)-f(y)|\le C|x-y|^{\alpha}$$

Case I: If f is holomorphic

on inside of C and C.

inside. $\tilde{f}(z)=f(z)$

outside $\tilde{f}(z)\equiv 0$

$$\tilde{f}_+(z_0)=f(z_0)=\frac{1}{2}f(z_0)+\frac{1}{2}f(z_0)$$

$$=\tilde{f}_p(z_0)+\frac{1}{2}f(z_0)$$

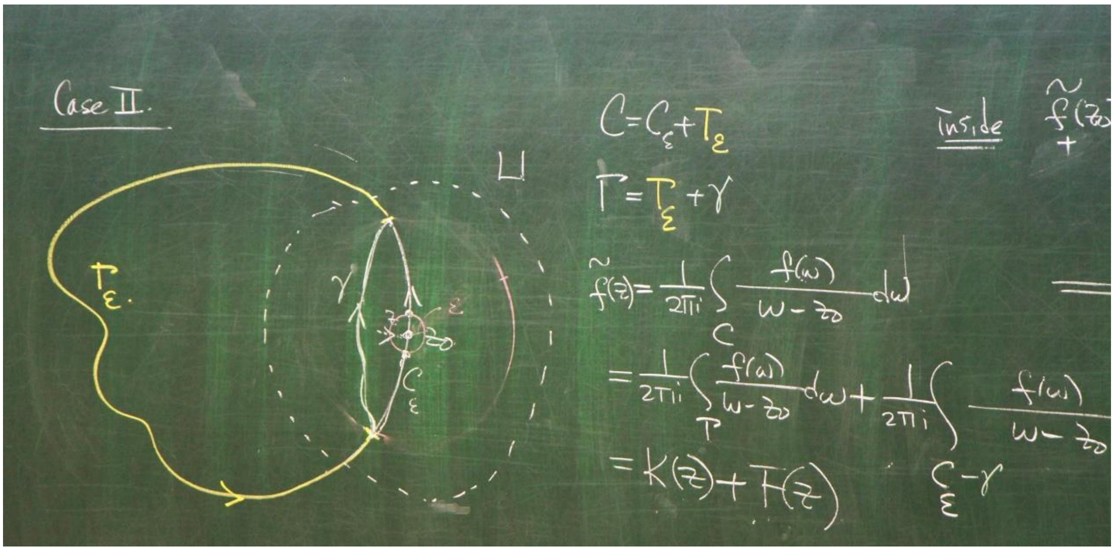
$$\tilde{f}_-(z_0)=0=\tilde{f}_p(z_0)-\frac{1}{2}f(z_0)$$

$\tilde{C}_\varepsilon=T_\varepsilon+C$

P.V.

$=\lim_{\varepsilon\to 0^+}$

$\frac{1}{2\pi i}\int_{\tilde{C}_\varepsilon}\frac{f(w)}{w-z}$



Case II.

$C=C_\varepsilon+T_\varepsilon$

$\Gamma=T_\varepsilon+\gamma$

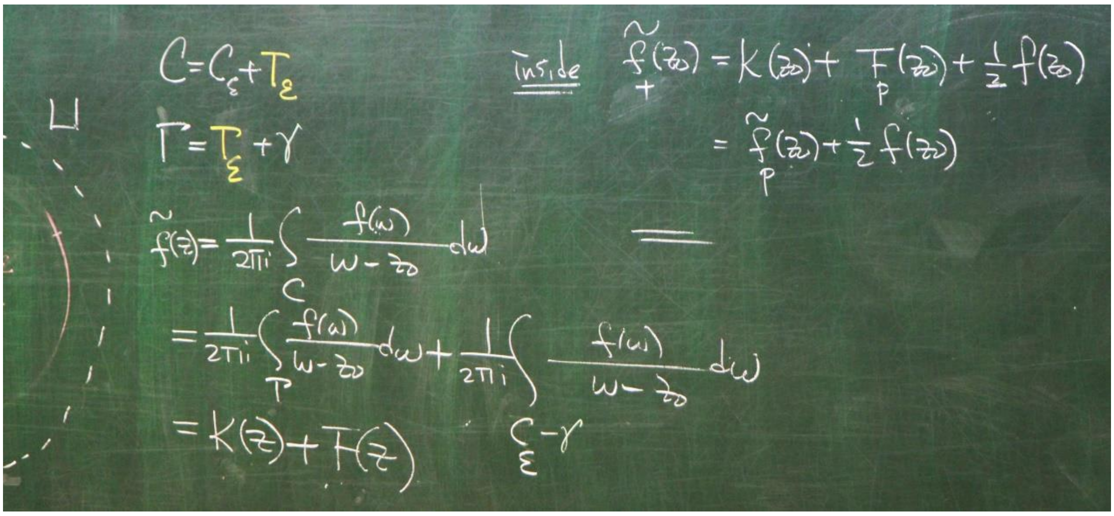
$$\tilde{f}(z)=\frac{1}{2\pi i}\int_C\frac{f(w)}{w-z_0}dw$$

$$=\frac{1}{2\pi i}\int_\Gamma\frac{f(w)}{w-z_0}dw+\frac{1}{2\pi i}\int_{C_\varepsilon-\gamma}\frac{f(w)}{w-z_0}$$

$$=K(z)+F(z)$$

inside $\tilde{f}_+(z_0)$

$=$



$C=C_\varepsilon+T_\varepsilon$

$\Gamma=T_\varepsilon+\gamma$

$$\tilde{f}(z)=\frac{1}{2\pi i}\int_C\frac{f(w)}{w-z_0}dw$$

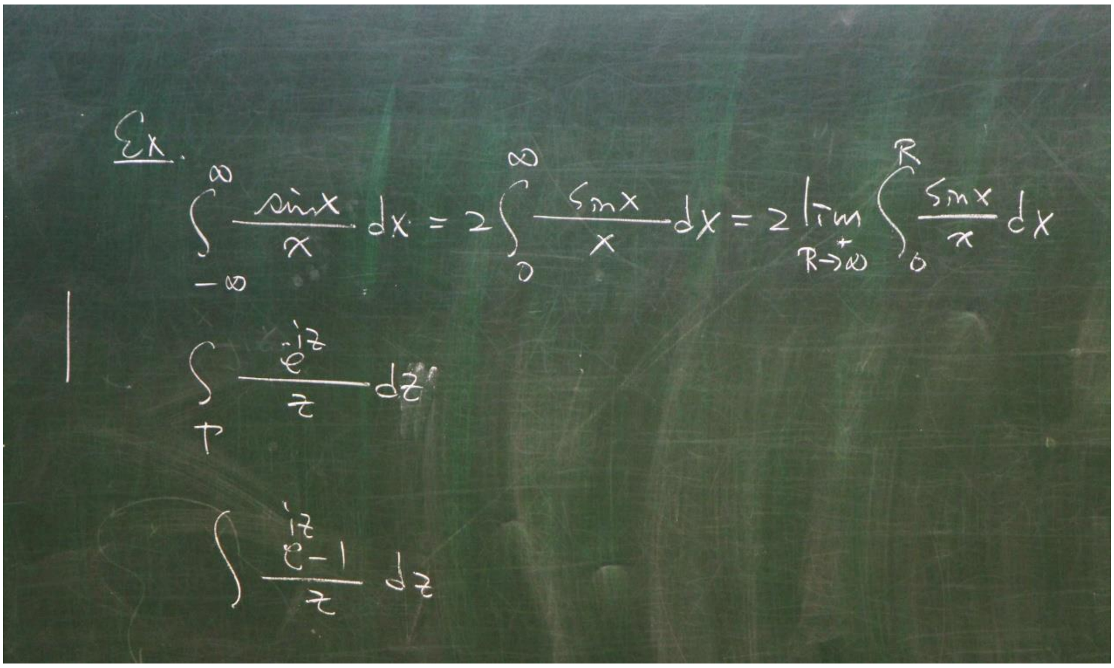
$$=\frac{1}{2\pi i}\int_\Gamma\frac{f(w)}{w-z_0}dw+\frac{1}{2\pi i}\int_{C_\varepsilon-\gamma}\frac{f(w)}{w-z_0}dw$$

$$=K(z)+F(z)$$

inside $\tilde{f}_+(z_0)=K(z_0)+F_p(z_0)+\frac{1}{2}f(z_0)$

$=\tilde{f}_p(z_0)+\frac{1}{2}f(z_0)$

$=$

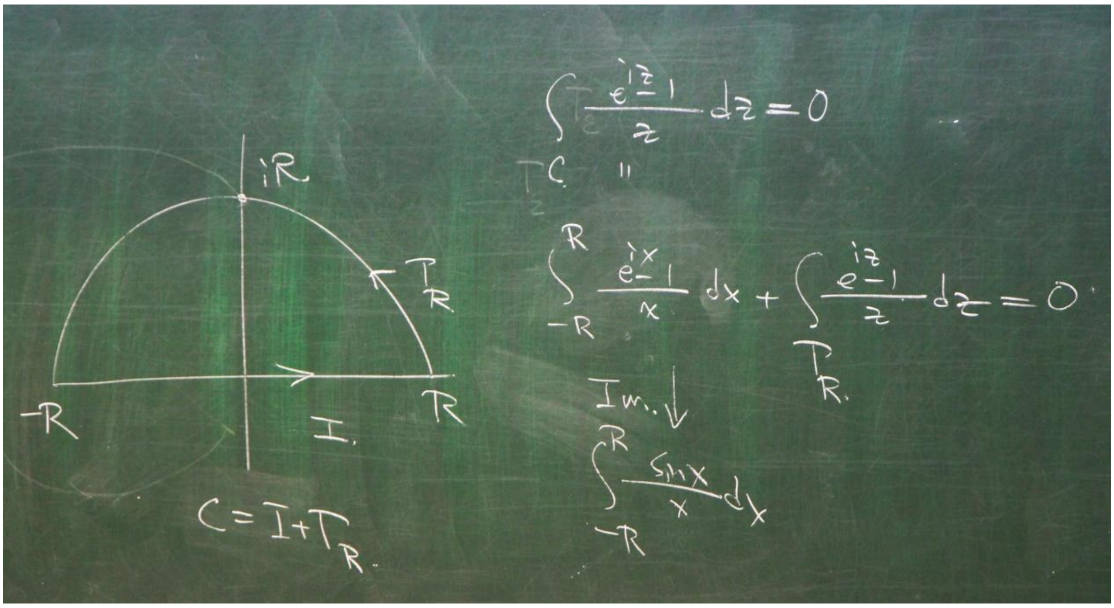


Ex.

$$\int_{-\infty}^{\infty}\frac{\sin x}{x}\,dx = 2\int_{0}^{\infty}\frac{\sin x}{x}\,dx = 2\lim_{R\to+\infty}\int_{0}^{R}\frac{\sin x}{x}\,dx$$

$$\int_{\Gamma}\frac{e^{iz}}{z}\,dz$$

$$\int \frac{e^{iz}-1}{z}\,dz$$



$C = I + \Gamma_R.$

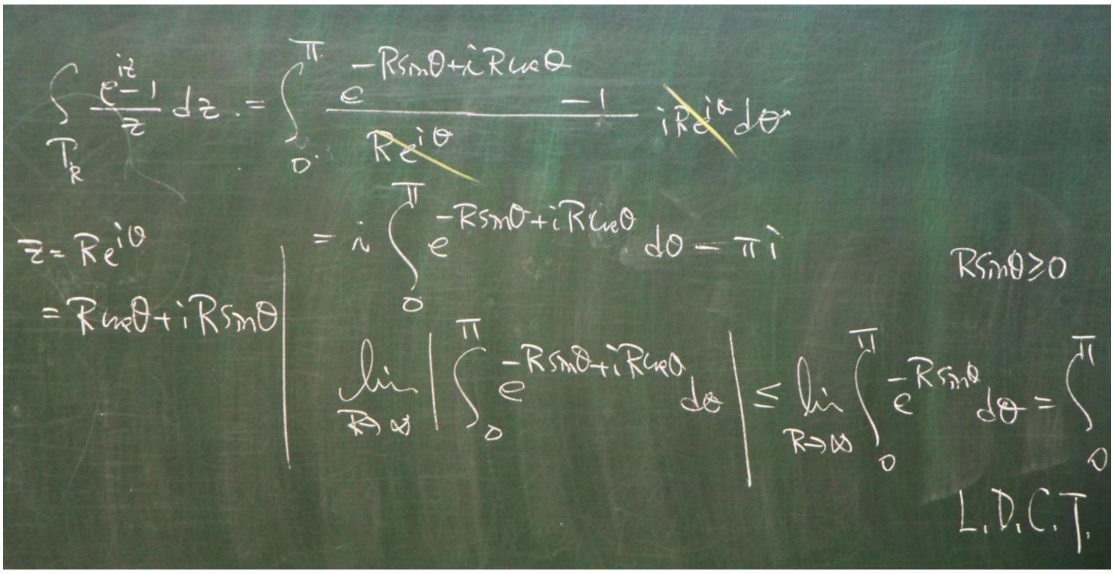
$$\int_{C}\frac{e^{iz}-1}{z}\,dz = 0$$

$$\parallel$$

$$\int_{-R}^{R}\frac{e^{ix}-1}{x}\,dx + \int_{\Gamma_R}\frac{e^{iz}-1}{z}\,dz = 0$$

$$\mathrm{Im}\downarrow$$

$$\int_{-R}^{R}\frac{\sin x}{x}\,dx$$



$$\int_{T_R} \frac{e^{iz}-1}{z}\,dz = \int_0^{\pi} \frac{e^{-R\sin\theta + iR\cos\theta} - 1}{R\cancel{e^{i\theta}}}\, i\cancel{Re^{i\theta}}\, d\theta$$

$z = Re^{i\theta}$

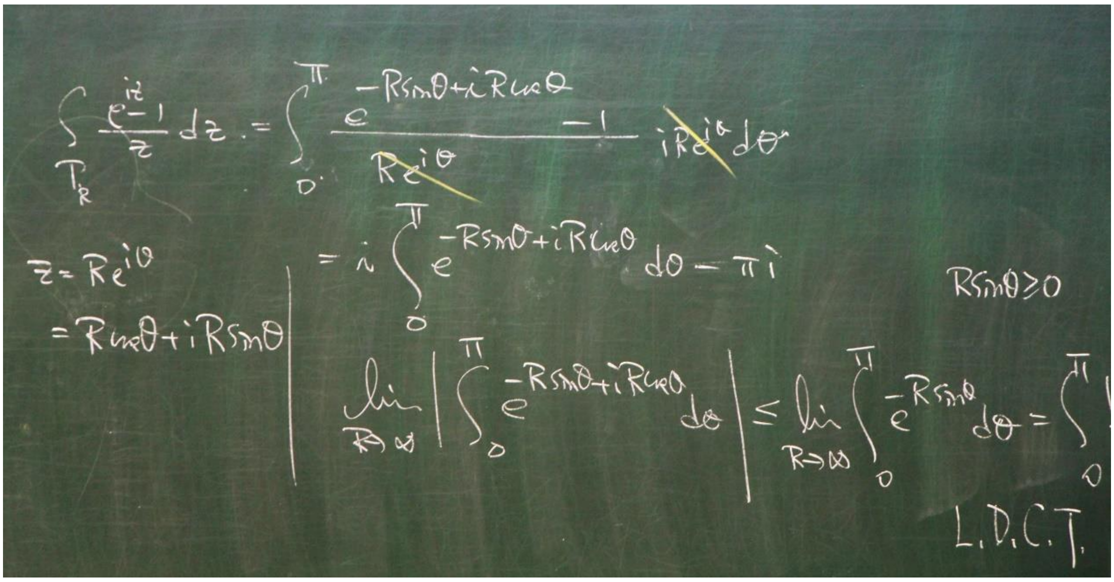
$= R\cos\theta + iR\sin\theta$

$$= i\int_0^{\pi} e^{-R\sin\theta + iR\cos\theta}\, d\theta - \pi i$$

$R\sin\theta \geq 0$

$$\lim_{R\to\infty}\left|\int_0^{\pi} e^{-R\sin\theta + iR\cos\theta}\, d\theta\right| \leq \lim_{R\to\infty}\int_0^{\pi} e^{-R\sin\theta}\, d\theta = \int_0^{\pi}$$

L.D.C.T.



$$\int_{T_R} \frac{e^{iz}-1}{z}\,dz = \int_0^{\pi} \frac{e^{-R\sin\theta + iR\cos\theta} - 1}{R\cancel{e^{i\theta}}}\, i\cancel{Re^{i\theta}}\, d\theta$$

$z = Re^{i\theta}$

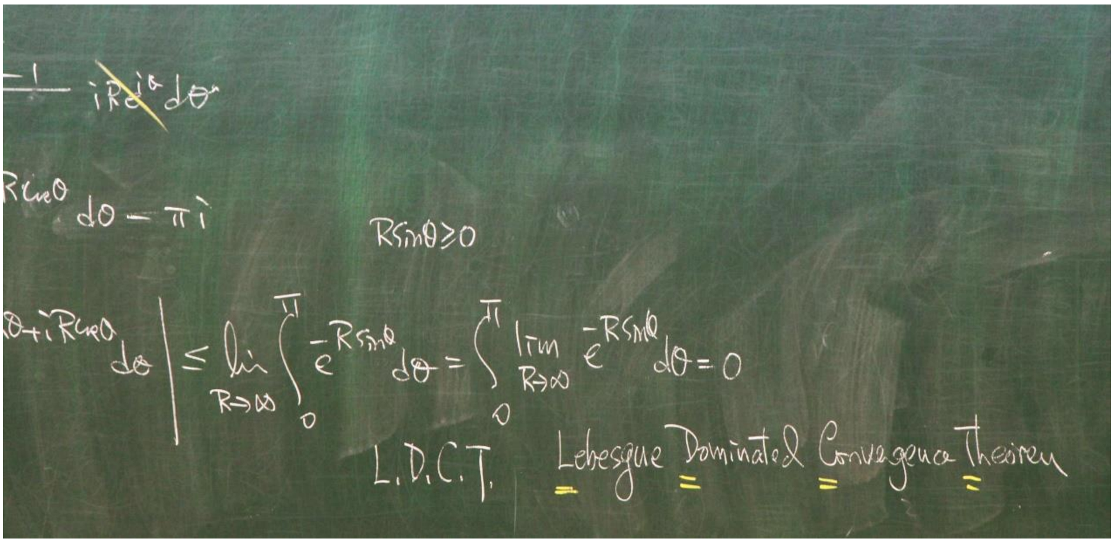
$= R\cos\theta + iR\sin\theta$

$$= i\int_0^{\pi} e^{-R\sin\theta + iR\cos\theta}\, d\theta - \pi i$$

$R\sin\theta \geq 0$

$$\lim_{R\to\infty}\left|\int_0^{\pi} e^{-R\sin\theta + iR\cos\theta}\, d\theta\right| \leq \lim_{R\to\infty}\int_0^{\pi} e^{-R\sin\theta}\, d\theta = \int_0^{\pi}$$

L.D.C.T.



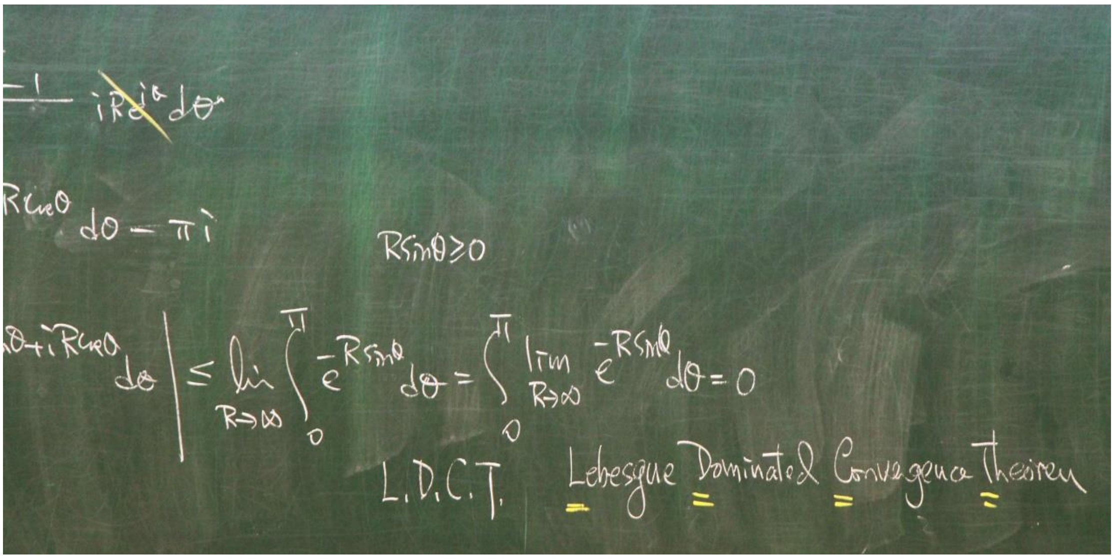
$-1 \quad i\cancel{Re^{i\theta}}\, d\theta$

$R\cos\theta \; d\theta - \pi i$

$R\sin\theta \geq 0$

$$\theta + iR\cos\theta\, d\theta \Big| \leq \lim_{R\to\infty}\int_0^{\pi} e^{-R\sin\theta}\, d\theta = \int_0^{\pi} \lim_{R\to\infty} e^{-R\sin\theta}\, d\theta = 0$$

L.D.C.T. | Lebesgue Dominated Convergence Theorem



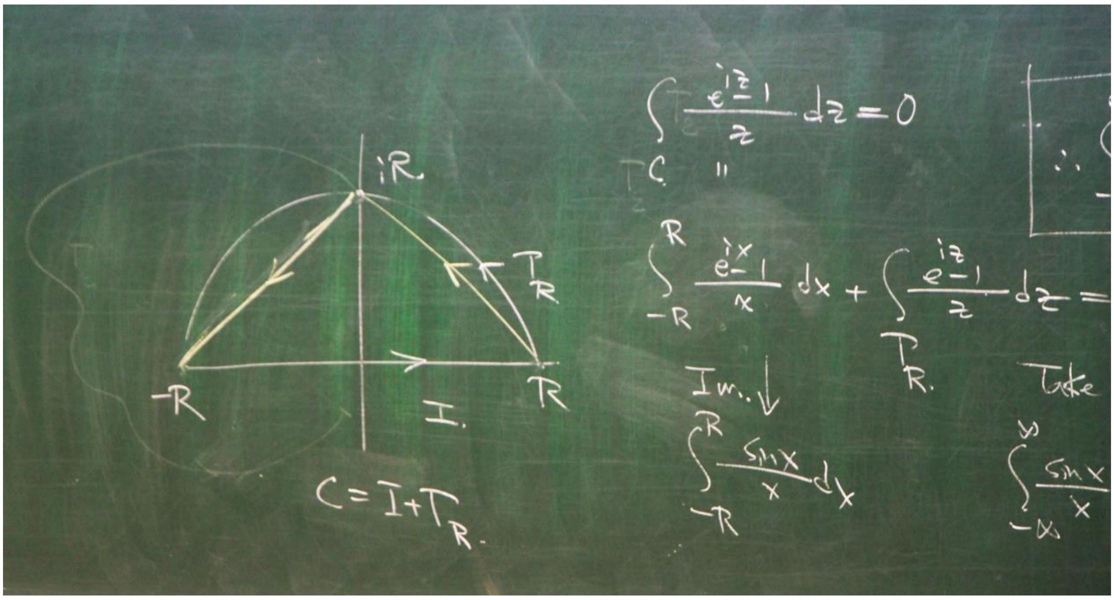
$\frac{1}{\ }\ iR\cancel{e^{i\theta}}\,d\theta$

$R\cos\theta\,d\theta - \pi i$

$R\sin\theta \geq 0$

$\theta + iR\cos\theta\,d\theta \Big| \le \lim_{R\to\infty}\int_0^{\pi} e^{-R\sin\theta}\,d\theta = \int_0^{\pi}\lim_{R\to\infty} e^{-R\sin\theta}\,d\theta = 0$

L.D.C.T. Lebesgue Dominated Convergence Theorem



$C = I + T_R$

$\int_C \frac{e^{iz}-1}{z}\,dz = 0$

$\qquad \shortparallel$

$\int_{-R}^{R}\frac{e^{ix}-1}{x}\,dx + \int_{T_R}\frac{e^{iz}-1}{z}\,dz =$

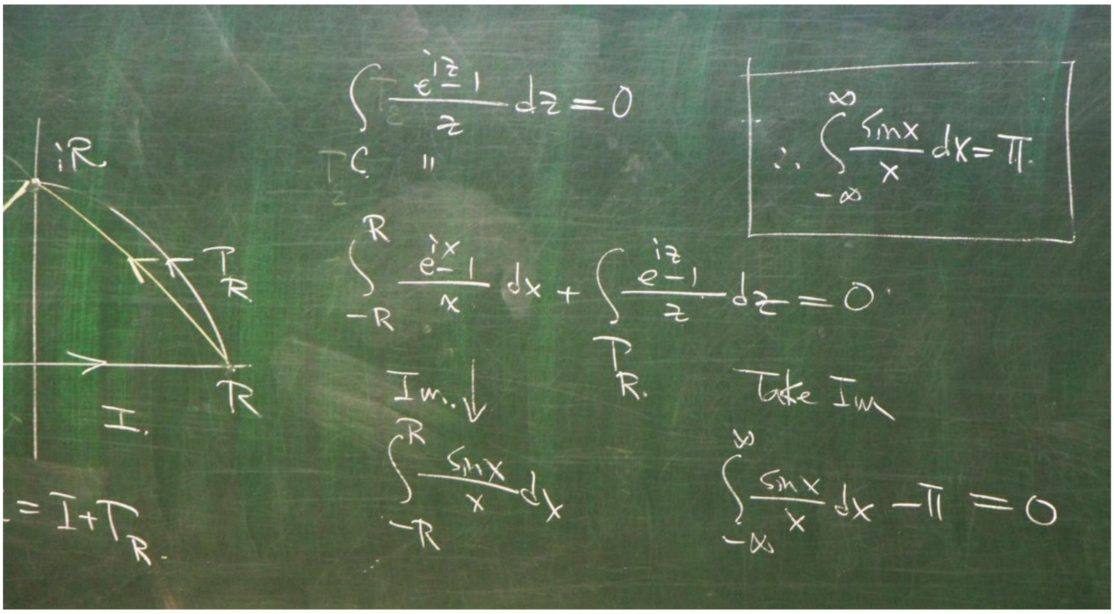
Im. $\downarrow$

$\int_{-R}^{R}\frac{\sin x}{x}\,dx$

Take

$\int_{-\infty}^{\infty}\frac{\sin x}{x}$

$\therefore$



$$\int_C \frac{e^{iz}-1}{z}\,dz = 0$$

$\|$

$$\int_{-R}^{R} \frac{e^{ix}-1}{x}\,dx + \int_{T_R} \frac{e^{iz}-1}{z}\,dz = 0.$$

Im ↓

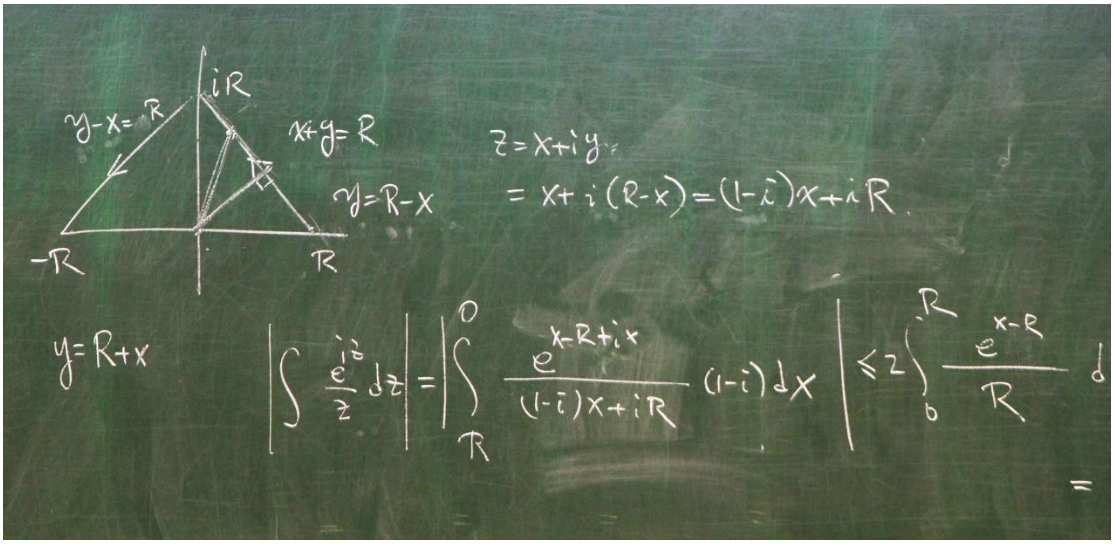
$$\int_{-R}^{R} \frac{\sin x}{x}\,dx$$

Take Im

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx - \pi = 0$$

$$\boxed{\therefore \int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \pi.}$$

$= I + T_R.$



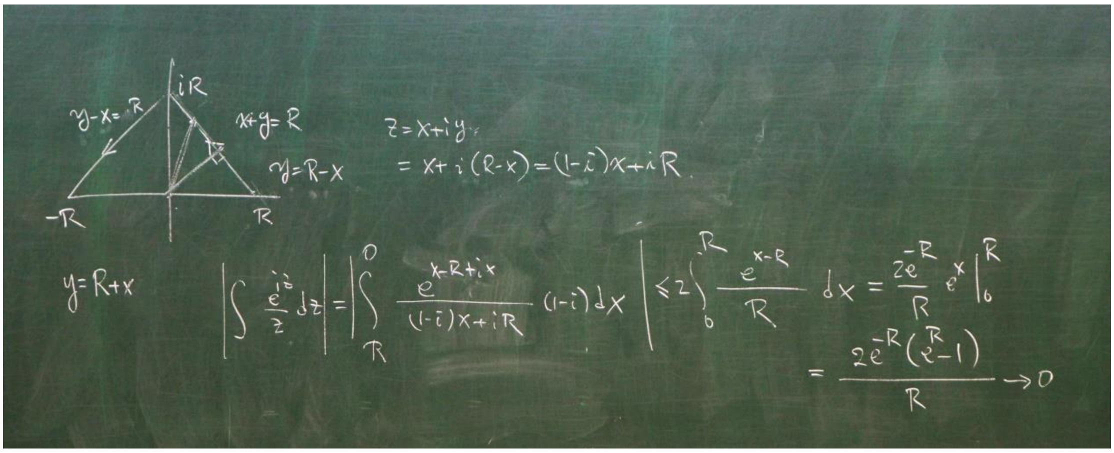
$y = R+x$

$z = x+iy$

$= x + i(R-x) = (1-i)x + iR.$

$$\left|\int \frac{e^{iz}}{z}\,dz\right| = \left|\int_R^0 \frac{e^{x-R+ix}}{(1-i)x+iR}(1-i)\,dx\right| \le 2\int_0^R \frac{e^{x-R}}{R}\,d$$

$=$



$y = R+x$

$z = x+iy$

$= x + i(R-x) = (1-i)x + iR.$

$$\left|\int \frac{e^{iz}}{z}\,dz\right| = \left|\int_R^0 \frac{e^{x-R+ix}}{(1-i)x+iR}(1-i)\,dx\right| \le 2\int_0^R \frac{e^{x-R}}{R}\,dx = \frac{2e^{-R}}{R}e^x\Big|_0^R$$

$$= \frac{2e^{-R}(e^R-1)}{R} \to 0$$